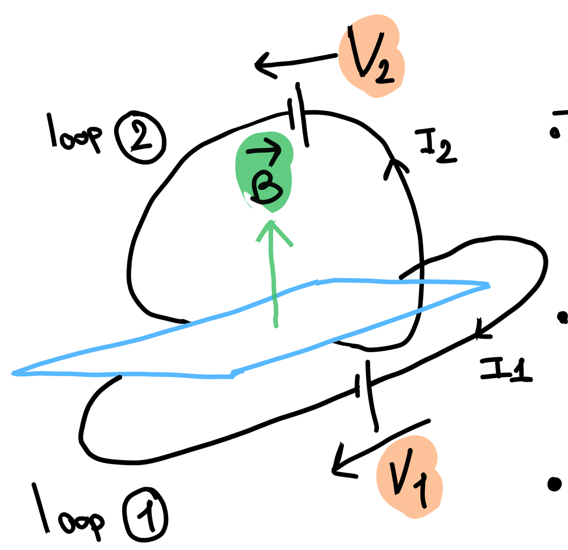
CHERN CHARACTER OF THE MULTILAYER FRACTIONAL QUANTUM HALL BUNDLE

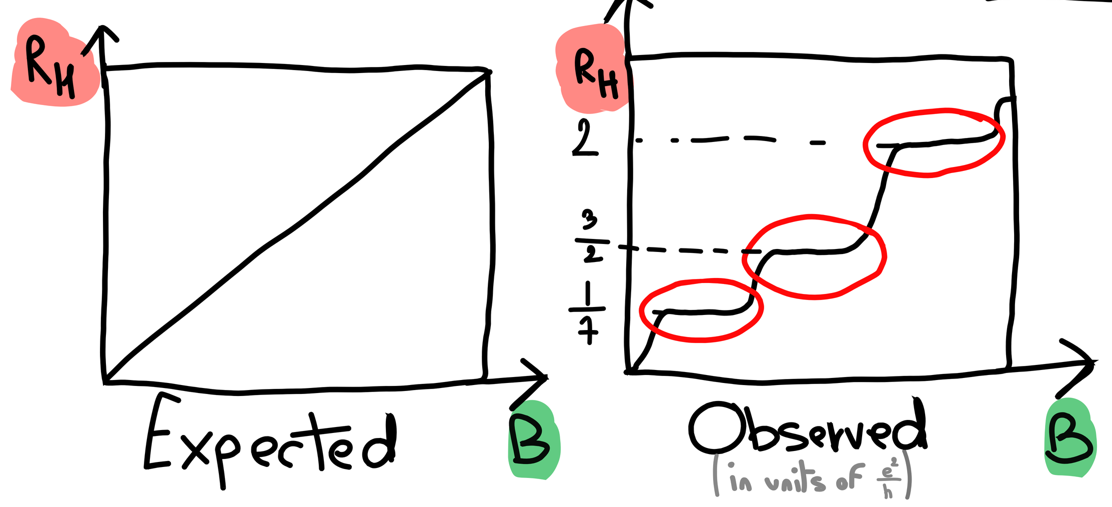
María Abad Aldonza (maria.abad-aldonza@uvsq.fr) - Florent Dupont (florent.dupont@math.unistra.fr)

1. What is the quantum Hall effect?



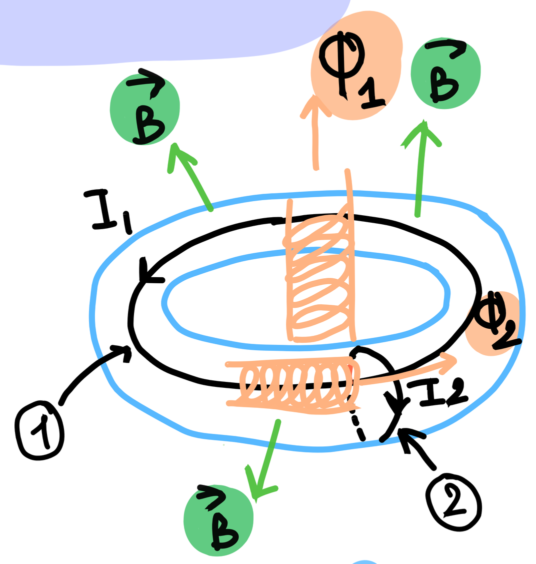
- Take a thin conductor plate and connect its edges as shown $\longleftrightarrow$
- Subject it to a constant magnetic field $\vec{B}$ $\longleftrightarrow$
- We can impose voltages V_1, V_2 and measure currents I_1, I_2. Electrons moving along loop 1 will be deviated by the magnetic field and induce a current along loop 2 $\longleftrightarrow$

- Prediction by classical mechanics: the transversal resistance $R_H = \frac{V_1}{I_2}$ is linear in B.
- Discovery in 1982: For some materials, R_H exhibits plateaus on which $\boxed{R_H \in \mathbb{Q}}$.



$\sigma_H = \frac{1}{R_H}$ is called the Hall conductance

2. Geometric Model



- Particles move on a compact orientable surface C of genus g with a riemannian metric
- The magnetic field is the curvature of a connection ∇ on a degree d hermitian line bundle $L \to C$.
- Currents are induced by changing the magnetic fluxes Φ_i going trough the cycles of C (changing the connection while keeping its curvature fixed)

N-particle quantum states are sections of $L^{\boxtimes N} \to C^N$ that satisfy the Shrödinger equation which has time dependent parameters (the connection ∇).

To get a conductance measurement, one must specify two cycles:
- one trough which the magnetic flux is changed
- one along which the current is measured

$\Rightarrow$ it can be seen as an object in cohomology.

The Hall conductance σ_H is the first chern class divided by rank of the vector bundle V of ground states.

Goal: Build V and compute $ch(V)$, from which we can recover σ_H

3. Why algebraic geometry?

- C is in particular an algebraic curve
- The connection ∇ defines a holomorphic structure on L by $\bar{\partial} = \nabla^{0,1}$
  One particle ground states are holomorphic sections of L
- Changing the connection $\Leftrightarrow$ Changing the holomorphic structure on L
  The parameter space is $Pic^d C$.

⚠ Problem: The Shrödinger equation is too difficult to solve.

Solution: Physicists make educated guesses on what ground states look like (ansatz).

We study the multilayer setting, where electrons move in $k \in \mathbb{N}$ copies of C.

In this case, the ansatz of physicists translated into geometrical language is that

$$\{\text{Multilayer states}\} = H^0\left(\prod_{i=1}^{k} S^{n_i}C, L^{\boxtimes \Sigma n_i}(-\Delta_K)\right)$$

$n_i = \#$ particles in layer i

Ground states
- are still holomorphic
- are (anti)-symmetric in every layer
- vanish to order K_{ij} when two particles in layer i and j meet

This is encoded in a symmetric matrix K and a divisor Δ_K

Locally those states have a factor $\prod_{ij}\prod_{\lambda\nu}(z_\lambda^i - z_\nu^j)^{K_{ij}}$.

4. The multilayer fractional quantum Hall bundle V

Ground states form a vector bundle V over $Pic^d C$ whose fiber above $[L] \in Pic^d C$ is $H^0\left(\prod_{i=1}^{k} S^{n_i}C, L^{\boxtimes \Sigma n_i}(-\Delta_K)\right)$.

It is built as a kind of Fourier-Mukai transform

$$\mathcal{L}_K := P^{\boxtimes \Sigma n_i} \otimes q^* \mathcal{O}(-\Delta_K) \qquad \boxed{V := p_* \mathcal{L}_K}$$

$$\begin{array}{ccccc} \mathcal{O}(-\Delta_K) & & \mathcal{L}_K & & V \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{i=1}^{k} S^{n_i}C & \xleftarrow{q} & \prod_{i=1}^{k} S^{n_i}C \times Pic^d C & \xrightarrow{p} & Pic^d C \end{array}$$

where, for a fixed $z_0 \in C$, $P \to C \times Pic^d C$ is the Poincaré line bundle defined by

$$\begin{cases} P_{|C \times [L]} \simeq L & \forall [L] \in Pic^d C \\ P_{|\{z_0\} \times Pic^d C} & \text{is trivial} \end{cases}$$

The Grothendieck-Riemann-Roch theorem applied to p gives:

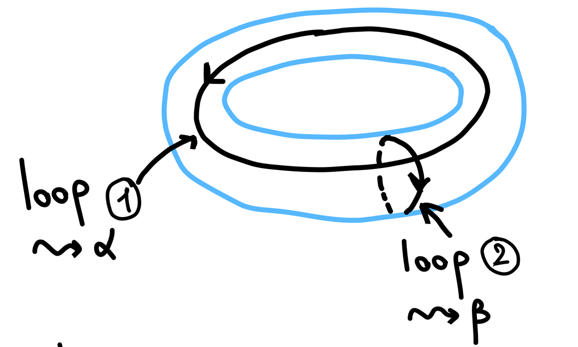
$$ch\left(\sum (-1)^i R^i p_* \mathcal{L}_K\right) = p_*\left(e^{c_1(\mathcal{L}_K)} \prod_{i=1}^{k} Td(S^{n_i}C)\right)$$

5. Computation of the pushforward

If $n > 2g-1$, $S^n C \xrightarrow{\pi} Pic^n C$ is the projectivisation of a vector bundle

$$\Rightarrow \mathcal{A}(S^n C) \simeq \mathcal{A}(Pic^n C)[\xi] \quad \text{where } \xi := c_1(\mathcal{O}(1)) \text{ on } S^n C$$

- Let $\Theta \subset Pic^n C$ be the theta divisor
  We have
  $$\pi_* \xi^{n-k} = \frac{\Theta^k}{k!} \quad \text{(Mattuck, 1961)}$$



- For the symplectic basis $(\alpha^s, \beta^s)_{1 \leq s \leq g}$ on $Pic^n C$ coming from the loops on C, $\Theta = \sum \alpha^s \wedge \beta^s$ in cohomology
- We wrote $e^{c_1(\mathcal{L}_K)} Td(\prod_i S^{n_i}C)$ in terms of these symplectic basis (one for each layer). Computing the pushforward becomes a combinatorial problem.

We use Wick's theorem, which states that in $R[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n] / \sim$

where
- R is a commutative ring with identity
- the α's, β's are ordered symbols
- $\sim$ makes the symbols anticommute

$$\forall A \in M_n(R), \quad e^{\alpha^T A \beta} = 1 + \ldots + \boxed{\det A}(\alpha_1\beta_1\alpha_2\beta_2 \cdots \alpha_n\beta_n) + \ldots$$

6. Main result

Lemma (M.AA-F.D)

When $K-I$ corresponds to a non-negative bilinear form, $\mathcal{L}_{K|\prod S^{n_i}C \times [L]} \otimes \omega^{-1}_{\prod S^{n_i}C}$ is ample $\forall [L]$, hence $\forall i > 0$, $R^i p_* \mathcal{L}_K = 0$ (Kodaira vanishing) and $\sum (-1)^i R^i p_* \mathcal{L}_K = V$.

Theorem (M.AA-F.D)

If $K-I$ corresponds to a non-negative bilinear form and $K\vec{n} + \vec{k}(g-1) = \vec{d}$,

particle numbers in each layer

diagonal entries of K

$$\vec{d} = \begin{pmatrix} d \\ \vdots \\ d \end{pmatrix}$$

$$\boxed{ch(V) = \underbrace{(\det K)^g}_{rank(V)} \exp\left(\underbrace{-|K^{-1}|}_{\text{conductance } \sigma_H}\Theta\right)}$$

where $|K^{-1}| = \sum_{ij} K^{-1}_{ij}$.

We also computed $ch(V)$ when $K\vec{n} + \vec{k}(g-1) \neq \vec{d}$.