Grothendieck-Riemann-Roch for Multilayer Quantum Hall Effect

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What is Quantum Hall Effect?

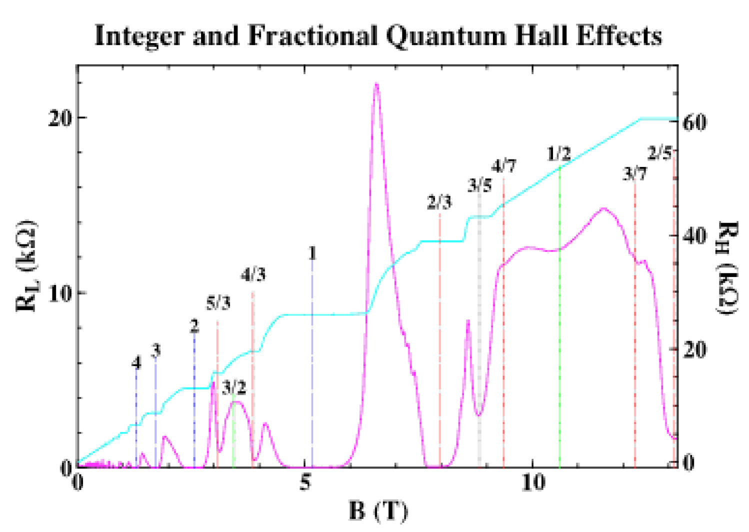


Figure 1. This graphic depicts both the Hall resistance R_H and the longitudinal resistance R_L with respect to the magnetic flux density B. Each plateau of R_H is marked with its corresponding filling factor ν. Image from [3]

Quantum Hall Effect is a phenomenon observed when electrons are forced to move in a low energy, 2D setting (for example, a very cold and thin metal plate) submitted to an intense orthogonal magnetic field.

Due to the magnetic field, an orthogonal voltage V_y appears in response to a longitudinal current with intensity I_x. The Hall resistance $R_H = {}^{V_y}/_{I_x}$, which is usually proportional to the flux of the magnetic field, is quantized when subjected to low temperatures.

R_H only takes values of the form ${}^{h}/_{e^2\nu}$, where e is the charge of the electron and h is the Planck constant. The variable ν is called the filling factor, it only takes certain rational values and is believed to measure the density of electrons occupying the lowest energy levels.

The "Ansatz first" approach

The Schrödinger equations that would describe the plateaus of QHE for rational filling factors have so many terms due to electron-electron interactions that they are impossible to solve analytically. Instead, educated guesses of the wavefunctions (ansatzs) are formulated based on experimental observations.

The vector spaces spanned by these wavefunctions are glued together taking into account time evolution to build a vector bundle V with a suitable hermitian connection.

The Hall conductance $\sigma_H = {}^{1}/_{R_H}$ is shown to be equal to the slope ${}^{\deg V}/_{\mathrm{rk}\, V}$ of this bundle. The fact that this is a topological invariant is meaningful from a physical point of view.

Laughlin states

Robert Laughlin pioneered this approach in his article [2], where he proposed n-particle ground state wavefunctions for the plateaus with $\nu = \frac{1}{m}$:

$$\psi_m(z_1, ..., z_n) = \prod(z_i - z_j)^m \exp(-\tfrac{eB}{4c\hbar}\sum|z_i|^2)$$

The associated Laughlin bundle was originally built over a sphere or a torus, but we can also consider Laughlin bundles over a general genus g compact Riemann surface.

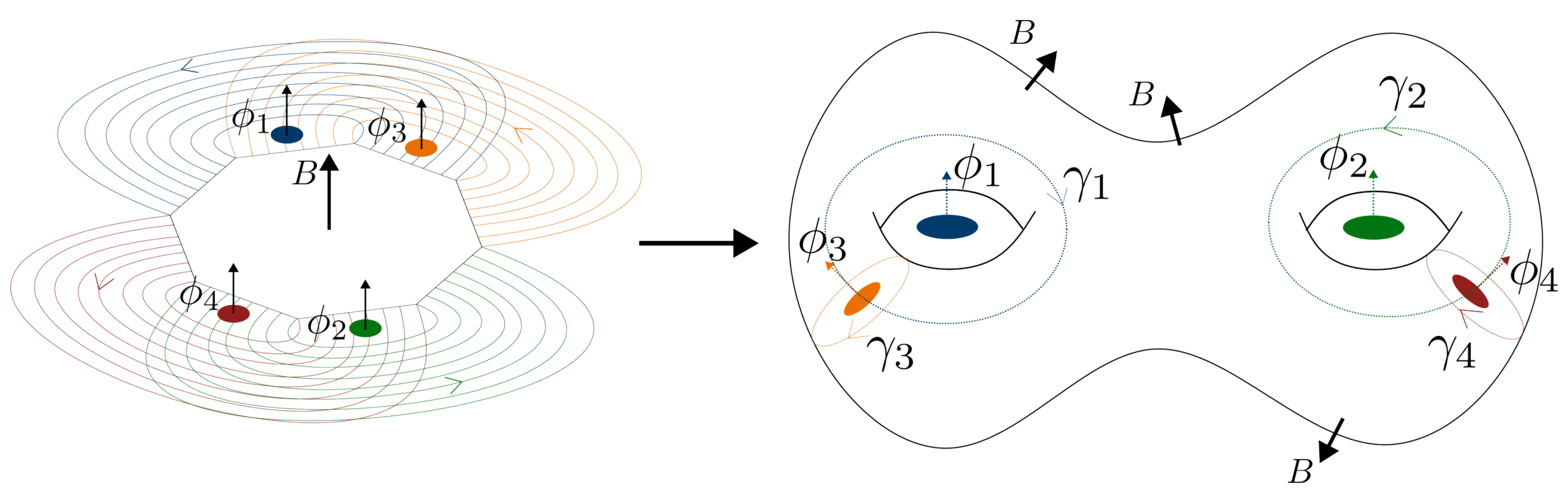


Figure 2. This is an example of what QHE over a genus 2 surface could look like. Image by F. Dupont

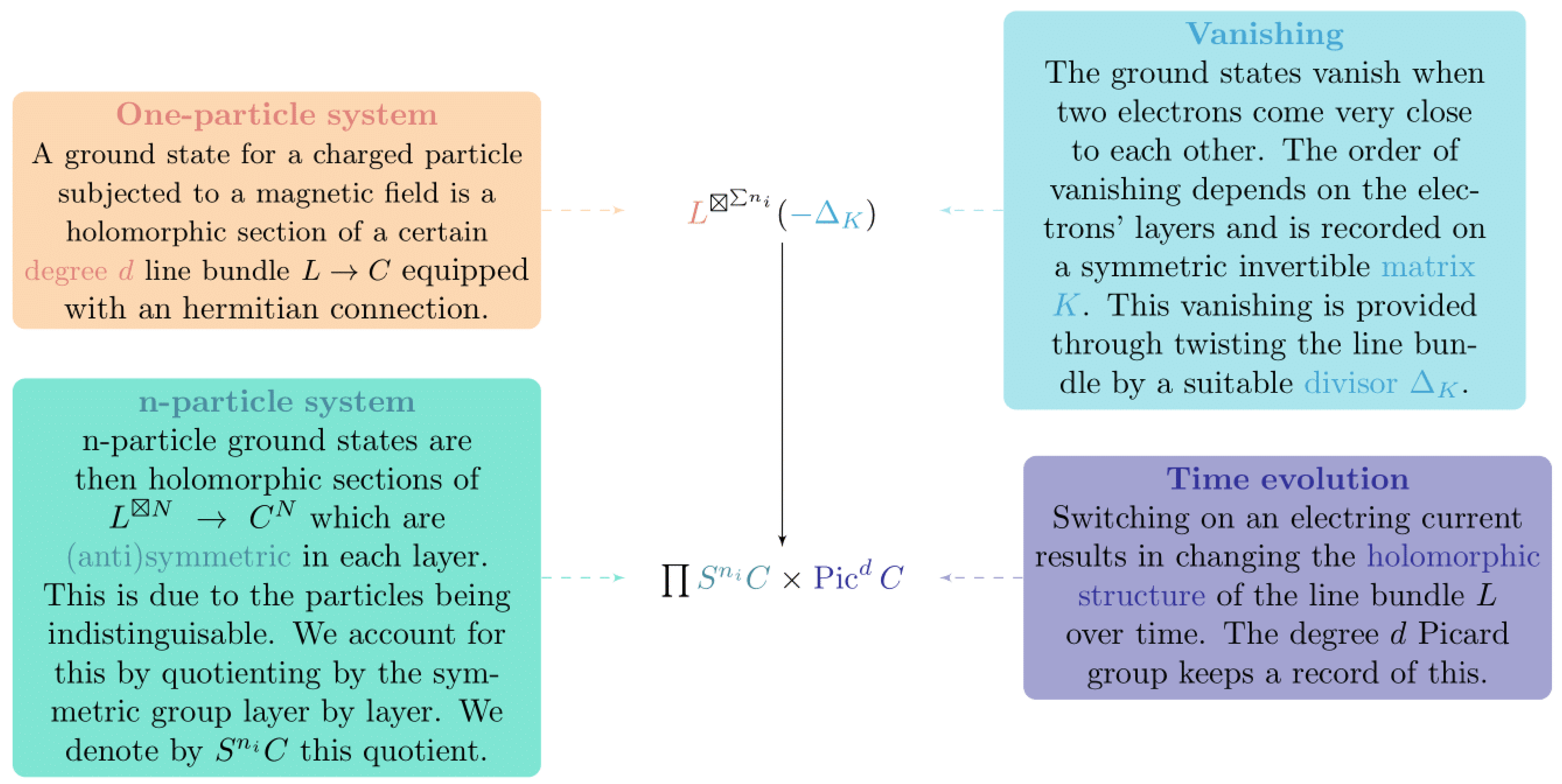
The "bundle first" approach

We could build a vector bundle V whose sections satisfy the vanishing conditions of Laughlin states without writing them out explicitly. Taking this approach, Klevstov and Zvonkine computed in [1] the Laughlin bundle's Chern character using algebraic geometric techniques. This approach has many advantages:

- It provides a rigorous computation of $\mathrm{rk}\, V = \mathrm{ch}_0(V)$ or, in other terms, an answer to the conjecture on whether the proposed ansatzes are the only suitable wavefunctions of the system.
- For $g > 1$, it provides other invariants on top of the Hall conductance. Namely, the higher Chern characters.
- Methods to explore QHE using ansatzes vary greatly from case to case. The "bundle first" approach seems easily transposable to other plateaux.

Multilayer Laughlin bundle

We applied the "bundle first" approach to the multilayer case. In this setting, the thin plate is composed of k layers, each identified with a genus g compact Riemann surface C. Multilayer Laughlin states are holomorphic section of the following line bundle L_K



Now consider the projection map $p : \prod S^{n_i}C \times \mathrm{Pic}^d C \to \mathrm{Pic}^d C$. The vector bundle $V = R^0 p_*(L_K)$ over $\mathrm{Pic}^d C$ is called the multilayer Laughlin bundle because its fibers are the vector spaces of "traditional" Laughlin functions for a given holomorphic structure on L.

Grothendieck-Riemann-Roch

The Grothendieck-Riemann-Roch formula

$$\mathrm{ch}\left(\sum(-1)^i R^i p_* L_K\right) = p_*(\mathrm{ch}(L_K)\,\mathrm{td}(p))$$

allows us to get an expression for the Chern character of the multilayer Laughlin bundle, provided that the higher derived pushforwards of L_K vanish. We proved this using Kodaira's vanishing theorem.

Since $\mathrm{ch}(L_K) = e^{c_1(L_K)}$, the next step is to compute the first Chern class of the line bundle L_K.

Mattuck's results on the cohomology of S^nC

Consider the map $\pi : S^nC \to \mathrm{Pic}^n C$ assigning to an unordered collection of points the corresponding divisor up to rational equivalence. Let $\Theta \subset \mathrm{Pic}^n C$ be the theta divisor and denote by θ its Poincaré dual. Mattuck proved in [4] that π is the projectivization of a vector bundle whose total Chern class is $e^{-\theta}$.

Fixing a point $z_0 \in C$, Mattuck's results allow us to express $c_1(L_K)$ in terms of the following cohomology classes:

- The class θ_i over $\mathrm{Pic}^{n_i} C$
- The Poincaré dual of the collection of points in $S^{n_i}C$ containing z_0, denoted by ξ_i
- A certain mixed class η_{ij} (resp. η_i) over $\mathrm{Pic}^{n_i} C \times \mathrm{Pic}^{n_j} C$ (resp. $\mathrm{Pic}^{n_i} C \times \mathrm{Pic}^d C$)

Proposition. *The first Chern class of the line bundle L_K equals*

$$c_1(L_K) = \sum_i(k_{ii}\theta_i - \eta_i) + \sum_{i<j} k_{ij}\eta_{ij} + \sum_i p_i\xi_i$$

where $p_i = d - \left(k_{ii}(n_i + g - 1) + \sum_j k_{ij}n_j\right)$ is known as the number of quasiholes on the i-th layer.

The final computation is carried out with the help of Wick's theorem.

Main result (simple case)

Theorem. *When there are no quasiholes (all $p_i = 0$), the Chern character of the multilayer Laughlin bundle is*

$$\mathrm{ch}(V) = \det(K)^g e^{-|K^{-1}|\theta}$$

This result agrees with what is known in QHE literature: For the multilayer model with no quasiholes, $\mathrm{rk}(L_K) = \det(K)^g$.

Main result (general case)

Theorem. *In the presence of quasiholes, the Chern character can be expressed in terms of minors of the matrix K:*

$$\mathrm{ch}(V) = \sum_{\substack{a_1,\dots,a_k \\ 0\le a_i\le p_i}} \left[\prod_{i=1}^{k}\binom{n_i - g + p_i}{p_i - a_i}\right] \sum_{\substack{I_1,\dots,I_g \\ \sharp(I\ni c)=a_c}} \prod_{l=1}^{g}\det(K_{I_l}) \sum_{m=0}^{g}\prod_{l=1}^{m}|K_{I_l}^{-1}|\frac{(-\theta)^m}{m!}$$

where

- *K_I is obtained from K by removing the rows and columns corresponding to indices outside of I*
- *$K^\sharp$ denotes the adjugate matrix*
- *$|K|$ stands for the sum of all coefficients in K*
- *$\sharp(I \ni c)$ stands for the number of subsets $I_\bullet \subset \{1, ..., k\}$ containing c.*

References

[1] Klevtsov, S., and Zvonkine, D. On laughlin vector bundles and fractional quantum hall effect.

[2] Laughlin, R. Anomalous quantum hall effect: an incompressible quantum fluid with fractionally charged excitations. *Physical Review Letters 50* (1983).

[3] Marcolli, M., and Mathai, V. Towards the fractional quantum hall effect: A noncommutative geometry perspective.

[4] Mattuck, A. Symmetric products and jacobians. *American Journal of Mathematics 83* (1961), 189.